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ON THE FIGURE OF MERCURY


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# ON THE FIGURE OF THE PLANET MERCURY

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Abstract: The figure of Mercury is estimated in terms of an isostatic form of equilibrium which tends to be controlled by the situation near perihelion passage at the 3:2 resonance spin rate. The ratios of the principal moments of inertia for Mercury are: (1) $(C-A)/C \geq 7 \times 10^{-5}$; (2) $(C-B)/C \geq 5 \times 10^{-5}$ and (3) $(B-A)/C \geq 2 \times 10^{-5}$. The thermal effect on Mercury's figure during solidification forces Mercury's rotation to be trapped in the 3:2 resonance lock as its spin rate is being slowed by tidal effects. It is shown that the process of trapping of Mercury has been naturally affected by the instantaneous solidification of Mercury into a shape with two thermal bulges, and that the two permanent thermal bulges stabilize the planet's rotation.

## Introduction:

In a recent report, Liu (1968) has shown that the axis of the two permanent thermal bulges on Mercury's surface approximately points to the Sun for about 60 earth days in each Mercurian day (176 earth days). Under this condition, the gravity gradient force due to the Sun can produce elongation along Mercury's perihelion axis. Therefore, it is possible that the shape of Mercury tends to be controlled by the situation near perihelion at the 3:2 resonance spin rate. The purpose of the present paper is to estimate the figure of Mercury under such circumstances of solidification. The effect of the thermal contraction or expansion of Mercury's figure on the planet's rotation is also investigated.

## Hydrostatic Theory

We consider a hydrostatic equilibrium shape of Mercury taking place near perihelion passage when Mercury's spin rate approaches 3:2 of its orbital mean motion. We take the axes at the center of Mercury, $\xi$ being measured away from the Sun at perihelion. Since $r_{26}$ is the mean distance between the center of mass of Mercury and the Sun when the true anomaly $|f| \lesssim 45^{\circ}$,

$$r_{26} = \frac{a(1-e^2)}{1+e\cos 26^{\circ}} \tag{1}$$

where a and e are semimajor axis and orbital eccentricity respectively. The gravitational potential at internal points $(\xi, \zeta, \eta)$ of Mercury due to the Sun is (Jeans, 1928)

$$V = \frac{GM_s}{r_{26}} - \frac{GM_s \xi}{r_{26}^2} + \frac{GM_s}{2r_{26}^3}(2\xi^2 - \zeta^2 - \eta^2) \tag{2}$$

where G is the gravitational constant and $M_s$ is the mass of the Sun.

Because Mercury's perihelion axis vibrates about the solar direction with an amplitude of the order of only 0.57 degrees in a period of about 30 earth days, the mean motion of each part of Mercury near perihelion passage is one of revolution with angular velocity $\sqrt{GM_s/r_{26}^3}$ about an axis perpendicular to the plane of the orbit and through the center of mass of Mercury and the Sun together, i.e., during this period Mercury very nearly keeps the same face toward the Sun.

The effect of this mean angular motion on the acceleration relative to Mercury is equivalent to that of a potential

$$W = \frac{GM_s}{2r_{26}^3}\left[(r_{26} + \xi)^2 + \zeta^2\right] \tag{3}$$

Hence, the total disturbing potential due to the Sun and the orbital motion is

$$\Delta(V+W) = \frac{GM_s}{2r_{26}^3}\left[(2\xi^2 - \zeta^2 - \eta^2) + (\xi^2 + \zeta^2)\right]. \tag{4}$$

This disturbing potential will affect the shape of Mercury. If Mercury could adjust its shape to form an equipotential surface (Jeffreys, 1959), the equation of Mercury's surface near perihelion passage would be

$$R = R_0\left[1 + \frac{5}{12}\cdot\frac{M_s}{M_m}\cdot\left(\frac{R_0}{r_{26}}\right)^3 \cdot \frac{7\xi^2 - 2\zeta^2 - 5\eta^2}{R_0^2}\right] \tag{5}$$

where $M_m$ is the mass of Mercury and $R_0$ is the spherical radius of Mercury.

The semi-axes X, Y and Z of the Mercurian Globe are:

$$X = \left[1 + \frac{35}{12}\cdot\frac{M_s}{M_m}\cdot\frac{R_0^3}{r_{26}^3}\right]R_0$$

$$Y = \left[1 - \frac{10}{12}\cdot\frac{M_s}{M_m}\cdot\frac{R_0^3}{r_{26}^3}\right]R_0 \tag{6}$$

$$Z = \left[1 - \frac{25}{12}\cdot\frac{M_s}{M_m}\cdot\frac{R_0^3}{r_{26}^3}\right]R_0$$

These formulae give increments of +5.6m, - 1.6m, and -4.0m respectively. If Mercury were completely adjusted to all disturbing potentials acting, the three axes would differ from the mean by those amounts.

Thus the ratios of the principal moments of inertia for Mercury are approximately,

$$\alpha_h = \frac{C-A}{C} = \frac{20}{4} \cdot \frac{M_s}{M_m} \cdot \frac{R_0^{\;3}}{r_{26}^{\;3}} = 4 \times 10^{-6}$$

$$\beta_h = \frac{C-B}{C} = \frac{5}{4} \cdot \frac{M_s}{M_m} \cdot \frac{R_0^{\;3}}{r_{26}^{\;3}} = 1 \times 10^{-6} \tag{7}$$

$$\lambda_h = \frac{B-A}{C} = \frac{15}{4} \cdot \frac{M_s}{M_m} \cdot \frac{R_0^{\;3}}{r_{26}^{\;3}} = 3 \times 10^{-6}$$

## Origin of Thermal Bulges

The liquid Mercurian Globe is assumed to be solidified in cooling at the 3:2 resonance spin rate. Since the heat loss by the liquid globe in solidifying must supply the actual heat loss by radiation from the surface, we obtain

$$L \cdot \frac{dm}{dt} = \sigma T^4 \cdot 4\pi R_0^{\;2} \tag{8}$$

where $L$ is the latent heat of fusion, $m$ is the mass that has solidified, $\sigma$ is the Stefan-Bolzmann constant and $T$ is the average surface temperatare. By integrating equation (8), the time required for solidification of Mercury is

$$t = \frac{(R_0^{\;3} - R_i^{\;3})\, L\, \rho}{3R_0^{\;2}\, \sigma\, T^4} \tag{9}$$

in which $\rho$ is the mean density and $R_0 - R_i$ is the depth of the solidifying shell.

For $L \leq 100$ Cal. $g.^{-1}$, $\rho \cong 5$ g. $Cm^{-3}$, $T = 1000$ deg., $\sigma = 5.7$ erg. $Cm.^{-2}$ $Sec.^{-1}$ $deg.^{-4}$ and $R_O = 2420$ Km., complete solidification of a shell with $R_O - R_i = 300$ Km. would not take more than 1000 years on the assumption that there was free access of heat to the surface by convection. If any part of the crust became so stiff as to stop convection, the supply of heat to the surface thereafter would be conduction and the surface would be cooled rapidly to the temperature maintained by solar heating (Liu, 1968). Solidification of the crust to a depth of 40 Km. would take less than 20 years. It seems reasonable to suppose that the material of the solidifying crust is sufficiently plastic to keep approximately the hydrostatic ellipticity. Therefore the time of formation of a crust, once it had started, would be exceedingly short and the hypothesis for instantaneous deformation of Mercury into a shape with two permanent thermal bulges is here justified.

Lower temperature at any part of the crust is accompanied by an increase of its radial thickness. Solidification at the poles and along the aphelion axis extends deeper than along the perihelion axis. Because of the unequal radial thickness of the shell and its unequal thermal expansion at different latitudes and longitudes Mercury should acquire the form corresponding to the isostatic equilibrium, i.e., the weight of the material in a column from the center to the surface must be the same in any direction. It has been found that such isostatic adjustment of the shell as a whole modifies the figure of Mercury from hydrostatic form $(\alpha_h, \beta_h, \lambda_h)$ to

$$\alpha \geq 7 \times 10^{-5},$$

$$\beta \geq 5 \times 10^{-5},$$

and $$\lambda \geq 2 \times 10^{-5}.$$

Based upon the last result, Liu and O'Keefe have developed the theory of rotation for Mercury (Liu and O'Keefe, 1965). Since the hydrostatic and thermoelastic nature of these bulges can be theoretically determined, it is suggested that the figure of Mercury may be now in isostatic equilibrium although it was in hydrostatic equilibrium before solidification.

Trapping

If the solidification of Mercury is connected with the transfer of radioactive elements toward the outside by repeated crystallization, the time needed for solidification of the whole of the shell in cooling would certainly be much longer than that just derived. For this reason, it has been investigated that the interval from the formation of a crust to the general solidification of the shell would have been millions of years. During this epoch of solidification in cooling, Mercury would rotate about its center of gravity with a time-dependent inertial tensor. It is necessary to consider at this critical time in what way the thermal expansion or contraction of Mercury's figure and the tidal action of the Sun would affect the planet's rotation.

The rotation of Mercury is governed by (Liu and O'Keefe, 1965)

$$\frac{d}{dt}\left[C_{(t)}\ \frac{d(f + \varphi)}{dt}\right] + \frac{3GM_s}{r^3}\left[B_{(t)} - A_{(t)}\right] \cos\varphi \sin\varphi = q \tag{10}$$

in which $f$ is the true anomaly, $\varphi$ is the angular displacement of the principal axis of $A_{(t)}$ from position vector of $r$, $q$ is the couple due to solar tides and $A_{(t)}$, $B_{(t)}$ and $C_{(t)}$ are principal moments of inertia which are functions of the time. During the earlier stage of this critical epoch, Mercury was subject to the conventional tidal action $q$ of the Sun, which, according to the fundamental theory of Darwin, should tend to retard Mercury's rotation. The retardational influence of the solar tides on Mercury's rotation is (Jeffreys, 1959)

$$-\ \frac{d(\dot{f} + \dot{\varphi})}{dt} = \frac{18}{5}\ \pi\ G\ \rho\ \mathrm{Sin}2\epsilon \cdot \left(\frac{M_m}{M_s}\right)^{-2}\left(\frac{R_o}{r}\right)^6 \tag{11}$$

where $2\epsilon$ is the lag of the conventional equilibrium tides. Equation (11) shows that the tidal effect is a very slow process for Mercury because its body is small. It has been estimated that the rotational period of Mercury would have increased by less than 0.3 days in an interval of the order of $10^7$ years. When Mercury solidified in cooling, the planet was also subject to contraction from loss of heat. The thermal contraction effect should tend to accelerate Mercury's rotation because it would produce a secular decrease of the value of $C_{(t)}$. A quantitative analysis of the amount of mechanical adjustment shows that the mean radius of Mercury would reduce about 7 Km. for an average cooling of order 300°C through depth down to about 300 Km. Under this

condition, the rotational period of Mercury after solidification of the whole shell would have decreased by about 0.3 day from that before solidification on the assumption that the Laplacian law of density was maintained. Thus the initiation of the tidal evolution is seen to be completely tied up with the influence of the thermal contraction during the epoch of solidification in cooling.

After solidification, Mercury's contraction from loss of heat must be exceedingly small. Also, the actual bodily tides must be much less than the conventional equilibrium tides because there can have been no shallow seas upon it. There is no positive ground for supposing any appreciable amount of bodily tidal friction in the Earth, and accordingly there is no strong reason to assume it in Mercury. Further, when Mercury's rotation approximately approached the 3:2 resonance condition the lag of the bodily tide in Mercury near every perihelion passage would be greatly decreased. For these reasons, Jeffreys (Jeffreys, 1959) has shown that the time required for an increase of 0.3 day in the present rotational period of Mercury would be one of the order of the $5 \times 10^9$ years estimated age of the solar system. The important question is, then, what would be happening if Mercury rotates completely free from tidal torque. Liu (Liu, 1966) has shown that the apparent circulational motion of Mercury at successive perihelia can be converted to a librational motion when Mercury rotates with any period within the range between $58.65 \pm 0.39$ days for $(B-A)/C = 5 \times 10^{-5}$. The application of the results obtained by Jeffreys and by Liu serves to predict that the axis of the two thermal bulges will continue to librate about the solar

direction at perihelion even deep into the future by cosmogonical standards.

It has been supposed until now that frictional dissipation by the bodily tide results in a retardation of Mercury's roation after solidification. This process is in the sense demanded by a loss of mechanical energy. However, the process of the thermal expansion of the two thermal bulges at perihelion passage is in the sense which increases the mechanical energy in the system, and this energy is extracted from the solar heat falling on the surface of the two thermal bulges by a thermal engine effect. The variation with time of the value $B_{(t)} - A_{(t)}$ due to solar heating will produce a thermodynamic tide on Mercury's thermal bulges. As the deviation of the axis of the two thermal bulges from the line of centers at perihelion increased due to the retardation effect by the bodily tide, it will cause the couple on Mercury's thermal bulges by the solar gravitation to be on an average positive. It is this couple that tends to increase the planet's rate of rotation while the bodily tidal couple tends to retard it. It has been found that the rate of supply of energy from solar heating and the dissipation of energy by the tidal friction are in an equilibrium condition with the periods of revolution and rotation locked in the 3:2 state. Therefore, the 3:2 agreement of periods that leads to the growth and oscillation of the thermodynamic tide on Mercury's thermal bulges is not fortuitous, but represents a final stable state because Mercury's rotation adjusts itself so as to maintain this state after solidification.

Hence, to explain the trapping of Mercury's rotational period into 3:2 resonance lock with its orbital period, we have shown that the balance of the influence of the tidal friction and thermal contraction provided the condition for Mercury to deform its shape with $(B-A)/C \geq 2 \times 10^{-5}$ during solidification, and that the initiation of the revolution of the bodily tide has been and will be completely tied up by the thermal engine effect.

## Conclusion

It has been shown that the whole mass of the liquid Mercury must have had the hydrostatic form to a considerable degree of accuracy at every perihelion passage when Mercury's spin rate approached 3:2 of its orbital mean motion. Instantaneous solidification distorted this form into a shape with two permanent thermal bulges which stabilized the planet's rotation.

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